QUANTUM RANDOM WALK APPROXIMATION ON LOCALLY COMPACT QUANTUM GROUPS

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Abstract. A natural scheme is established for the approximation of quantum Lévy processes on locally compact quantum groups by quantum random walks. We work in the somewhat broader context of discrete approximations of completely positive quantum stochastic convolution cocycles on C^*-bialgebras.

Introduction

In [LS_2] we developed a theory of quantum stochastic convolution cocycles on counital multiplier C^*-bialgebras, extending the *algebraic* theory of quantum Lévy processes created by Schürmann and coworkers (see [Sch] and references therein) and the *topological* theory of quantum stochastic convolution cocycles on compact quantum groups and operator space coalgebras developed by the authors ([LS_1]). Here we apply the results of [LS_2] to introduce and analyse a straightforward scheme for the approximation of such cocycles by quantum random walks. In particular we obtain results on Markov-regular quantum Lévy processes on locally compact quantum semigroups, extending and strengthening results in [FrS] for the compact case. Our analysis exploits a recent approximation theorem of Belton ([Be_2]), which extends that of [Sah] (used in [FrS]). The approximation scheme closely mirrors the way in which Picard iteration operates in the construction of solutions of quantum stochastic differential equations (see [L]).

The study of quantum random walks on quantum groups was intitiated by Biane in the early nineties (starting from [Bia]). Some combinatorial, probabilistic and physical interpretations can be found in Chapter 5 of [Maj]. Recent work has concentrated on discrete quantum groups and the development of (Poisson and Martin) boundary theory for quantum random walks (see [NeT] and references therein). Random walks of the type considered here and in [FrS] are discussed in [FrG] in the context of finite quantum groups. For *standard* quantum stochastic cocycles on operator algebras and operator spaces (see [L], and references therein), quantum random walk approximation ([LiP], [Sah], [Be_2]) has seen recent applications in the probability theory and mathematical physics literature (e.g. [AtP], [BvH]).

1. Preliminaries

In this section we briefly recall some definitions and relevant facts about strict maps and their extensions, matrix spaces over an operator space, structure maps with respect to a character on a C^*-algebra, multiplier C^*-bialgebras and quantum stochastic convolution cocycles; we refer to [LS_2] for a detailed account.

General notations. The multiplier algebra of a C^*-algebra A is denoted by $M(\mathsf{A})$ (note that in [LS_2] $\tilde{\mathsf{A}}$ was used). The symbols $\underline{\otimes}$, $\otimes$ and $\overline{\otimes}$ are used respectively for linear/algebraic, spatial/minimal and ultraweak tensor products, of spaces

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and respecively, linear, completely bounded and ultraweakly continuous completely bounded maps. For a subset S of a vector space V we denote its linear span by $\operatorname{Lin} S$. For a Hilbert space h, we have the ampliation

$$\iota_{\mathsf{h}} : B(\mathsf{H};\mathsf{K}) \to B(\mathsf{H}\otimes\mathsf{h};\mathsf{K}\otimes\mathsf{h}),\ T \mapsto T\otimes I_{\mathsf{h}}$$

where context determines the Hilbert spaces H and K; we also use the notations

$$\widehat{\mathsf{h}} := \mathbb{C}\oplus\mathsf{h},\quad \widehat{c} := \begin{pmatrix}1\\ c\end{pmatrix} \text{ for } c\in\mathsf{h} \text{ and } \Delta^{\mathrm{QS}} := P_{\{0\}\oplus\mathsf{h}} = \begin{bmatrix}0 & \\ & I_{\mathsf{h}}\end{bmatrix} \in B(\widehat{\mathsf{h}}) \tag{1.1}$$

(the superscript QS is there to avoid confusion with coproducts).

Strict maps and their extensions. If $\mathsf{A}_1, \mathsf{A}_2$ are C^*-algebras then a map $\varphi : \mathsf{A}_1 \to M(\mathsf{A}_2)$ is called strict if it is bounded and continuous in the strict topology on bounded subsets. The space of all such maps is denoted $B_\beta(\mathsf{A}_1; M(\mathsf{A}_2))$. Each map $\phi \in B_\beta(\mathsf{A}_1; M(\mathsf{A}_2))$ has a unique strict extension $\tilde{\varphi} : M(\mathsf{A}_1) \to M(\mathsf{A}_2)$. This allows the natural composition operation: if $\psi \in B_\beta(\mathsf{A}_2; M(\mathsf{A}_3))$ for another C^*-algebra A_3 we define $\psi\circ\varphi := \tilde{\psi}\circ\varphi$. A map $\varphi \in B_\beta(\mathsf{A}_1; M(\mathsf{A}_2))$ is called *preunital* if its strict extension is unital. Thus, when φ is *-homomorphic, preunital is equivalent to nondegenerate ([Lan]).

It is shown in Section 1 of [LS$_2$] that every completely bounded map from a C^*-algebra to the algebra of all bounded operators on a Hilbert space (understood as the multiplier algebra of the algebra of compact operators) is automatically strict.

Matrix spaces. For an operator space V in $B(\mathsf{H};\mathsf{K})$ and full operator space $B = B(\mathsf{h};\mathsf{k})$, the (h,k)-*matrix space over* V, denoted $\mathsf{V}\otimes_{\mathrm{M}} B$, is

$$\{A \in B(\mathsf{H}\otimes\mathsf{h},\mathsf{K}\otimes\mathsf{k}) : \forall_{\omega\in B_*}\ (\mathrm{id}_{B(\mathsf{H};\mathsf{K})}\,\overline{\otimes}\,\omega)(A) \in \mathsf{V}\}.$$

It is an operator space lying between $\mathsf{V}\underline{\otimes} B$ and $\overline{\mathsf{V}}\,\overline{\otimes}\, B$ which is equal to the latter when V is ultraweakly closed. For any map $\varphi \in CB(\mathsf{V}_1;\mathsf{V}_2)$, between operator spaces, the map $\varphi\,\underline{\otimes}\,\mathrm{id}_B$ extends uniquely to a completely bounded map $\varphi\otimes_{\mathrm{M}}\mathrm{id}_B : \mathsf{V}_1\otimes_{\mathrm{M}} B \to \mathsf{V}_2\otimes_{\mathrm{M}} B$ ([LiW]). This construction is compatible with strict tensor products and strict extension, in the sense described in Section 1 of [LS$_2$].

χ-structure maps. Let (A,χ) be a C^*-algebra with character. A χ-*structure map on* (A,χ) is a linear map $\varphi : \mathsf{A} \to B(\widehat{\mathsf{h}})$, for some Hilbert space h, satisfying

$$\varphi(a^*b) = \varphi(a)^*\chi(b) + \chi(a)^*\varphi(b) + \varphi(a)^*\Delta^{\mathrm{QS}}\varphi(b),$$

where Δ^{QS} is given by (1.1). The following automatic implementability result, which is established in [LS$_1$], is key.

Theorem 1.1. *Let (A,χ) be a C^*-algebra with character and let φ be a linear map $\mathsf{A}\to B(\widehat{\mathsf{h}})$, for some Hilbert space h. Then the following are equivalent.*

(i) *φ is a χ-structure map.*
(ii) *φ is* implemented *by a pair (π,ξ) consisting of a *-homomorphism $\pi : \mathsf{A}\to B(\mathsf{h})$ and vector $\xi\in\mathsf{h}$, that is φ has block matrix form*

$$a \mapsto \begin{bmatrix}\gamma(a) & \langle\xi|\nu(a)\\ \nu(a)|\xi\rangle & \nu(a)\end{bmatrix} \text{ where } \gamma := \omega_\xi\circ\nu \text{ for } \nu := \pi - \iota_{\mathsf{h}}\circ\chi. \tag{1.2}$$

Moreover, if φ is a χ-structure map with such a block matrix form then it is necessarily strict, and π is nondegenerate if and only if $\widetilde{\varphi}(1) = 0$.

Multiplier C^*-bialgebras. A (*multiplier*) *C^*-bialgebra* is a C^*-algebra B with *coproduct*, that is a nondegenerate *-homomorphism $\Delta : \mathsf{B} \to M(\mathsf{B} \otimes \mathsf{B})$ satisfying the coassociativity conditions

$$(\mathrm{id}_{\mathsf{B}} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_{\mathsf{B}}) \circ \Delta.$$

A *counit* for (B, Δ) is a character ϵ on B satisfying the counital property:

$$(\mathrm{id}_{\mathsf{B}} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \mathrm{id}_{\mathsf{B}}) \circ \Delta = \mathrm{id}_{\mathsf{B}}.$$

Examples of counital C^*-bialgebras include locally compact quantum groups in the universal setting ([Kus]), in particular all coamenable locally compact quantum groups are included.

Let B be a C^*-bialgebra. The *convolute* of maps $\phi_1 \in \mathrm{Lin}\, CP_\beta(\mathsf{B}; M(\mathsf{A}_1))$ and $\phi_2 \in \mathrm{Lin}\, CP_\beta(\mathsf{B}; M(\mathsf{A}_2))$ for C^*-algebras A_1 and A_2 is defined by composition of strict maps:

$$\phi_1 \star \phi_2 = (\phi_1 \otimes \phi_2) \circ \Delta \in \mathrm{Lin}\, CP_\beta(\mathsf{B}; M(\mathsf{A}_1 \otimes \mathsf{A}_2));$$

the same notation is used for its strict extension. The convolution operation is easily seen to be associative.

Note that, by automatic strictness, maps $\varphi_1 \in CB(\mathsf{B}; B(\mathsf{h}_1))$ and $\varphi_2 \in CB(\mathsf{B}; B(\mathsf{h}_2))$, for Hilbert spaces h_1 and h_2, may be convolved:

$$\varphi_1 \star \varphi_2 \in CB(\mathsf{B}; B(\mathsf{h}_1 \otimes \mathsf{h}_2)). \tag{1.3}$$

Quantum stochastic convolution cocycles. *We now fix, for the rest of the paper*, a complex Hilbert space k refered to as the *noise dimension space* and a counital C^*-bialgebra B.

For $0 \le r < t \le \infty$ we let $\mathcal{F}_{[r,t[}$ denote the symmetric Fock space over $L^2([r,t[; \mathsf{k})$ and write $I_{[r,t[}$ for the identity operator on $\mathcal{F}_{[r,t[}$ and $\mathcal{F}$ for $\mathcal{F}_{[0,\infty[}$. Also let $\mathcal{E}$ denote the linear span of $\{\varepsilon(g) : g \in L^2(\mathbb{R}_+; \mathsf{k})\}$, where $\varepsilon(g)$ denotes the exponential vector $\big((n!)^{-\frac{1}{2}} g^{\otimes n}\big)_{n \ge 0}$ in $\mathcal{F}$.

For $\varphi \in CB(\mathsf{B}; B(\widehat{\mathsf{k}}))$, the coalgebraic QS differential equation

$$dl_t = l_t \star d\Lambda_\varphi(t), \quad l_0 = \iota_{\mathcal{F}} \circ \epsilon.$$

has a unique *form* solution, denoted l^φ; it is actually a *strong* solution. The process l^φ is a QS convolution cocycle on B; moreover, conversely any Markov-regular, completely positive, contractive, QS convolution cocycle on B is of the form l^φ for a unique $\varphi \in CB(\mathsf{B}; B(\widehat{\mathsf{k}}))$. For completely bounded processes the cocycle relation reads as follows (after some natural identifications are made):

$$l_{s+t} = l_s \star \big(\sigma_s \circ l_t\big), \quad l_0 = \iota_{\mathcal{F}} \circ \epsilon, \quad s, t \in \mathbb{R}_+,$$

where $(\sigma_s)_{s \ge 0}$ is the semigroup of right shifts on $B(\mathcal{F})$. *Markov-regularity* means that each of the associated convolution semigroups of the cocycle is norm-continuous. In this situation, the map φ is referred to as the *stochastic generator* of the QS convolution cocycle. A QS convolution cocycle l is said to be completely positive, preunital, *-homomorphic, etc. if each l_t has that property. Generators of such cocycles are characterised in Theorem 5.2 of [LS$_2$].

2. Approximation by discrete evolutions

We now show that any Markov-regular, completely positive, contractive QS convolution cocycle on B may be approximated in a strong sense by discrete completely positive evolutions, and that the discrete evolutions may be chosen to be *-homomorphic and/or preunital, if the cocycle is.

Belton's condition ([Be$_2$]) for discrete approximation of standard Markov-regular QS cocycles ([L]) nicely translates to the convolution context using the techniques

developed in [LS$_2$]. We show this first. Denote by $\Xi_n^{(h)}$ $(h > 0, n \in \mathbb{N})$ the injective *-homomorphism

$$B(\widehat{\mathsf{k}}^{\otimes n}) = B(\widehat{\mathsf{k}})^{\overline{\otimes}\, n} \to B(\mathcal{F}_{[0,hn[}) \otimes I_{[hn,\infty[} = \Big(\overline{\bigotimes}_{j=1}^{n} B(\mathcal{F}_{[(j-1)h,jh[})\Big) \otimes I_{[hn,\infty[}$$

arising from the discretisation of Fock space ([AtP], [Be$_1$]). Thus

$$\Xi_n^{(h)} : A \mapsto D_n^{(h)} A D_n^{(h)*} \otimes I_{[hn,\infty[}$$

where

$$D_n^{(h)} := \bigotimes_{j=1}^{n} D_{n,j}^{(h)}, \text{ for the isometries}$$

$$D_{n,j}^{(h)} : \widehat{\mathsf{k}} \mapsto \mathcal{F}_{[(j-1)h,jh[}, \ \begin{pmatrix} z \\ c \end{pmatrix} \mapsto \big(z, h^{-1/2} c_{[(j-1)h,jh[}, 0, 0, \cdots\big).$$

Also write $\Xi_{n,\varepsilon}^{(h)}$ for the completely bounded map

$$B(\widehat{\mathsf{k}}^{\otimes n}) \to |\mathcal{F}\rangle, \quad A \mapsto \Xi_n^{(h)}(A)|\varepsilon\rangle, \quad \text{ where } h > 0, n \in \mathbb{N} \text{ and } \varepsilon \in \mathcal{E}.$$

For a map $\Psi \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})\big)$, in which V is a concrete operator space, its *composition iterates* $(\Psi^{\bullet n})_{n \in \mathbb{Z}_+}$ are defined recursively by

$$\Psi^{\bullet 0} := \mathrm{id}_{\mathsf{V}}, \quad \Psi^{\bullet n} := \big(\Psi^{\bullet (n-1)} \otimes_{\mathrm{M}} \mathrm{id}_{B(\widehat{\mathsf{k}})}\big) \circ \Psi \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}^{\otimes n})\big), \quad n \in \mathbb{N}.$$

Similarly, for a map $\psi \in CB\big(\mathsf{B}; B(\widehat{\mathsf{k}})\big)$, its *convolution iterates* $(\psi^{\star n})_{n \in \mathbb{Z}_+}$ are defined by

$$\psi^{\star 0} := \epsilon, \quad \psi^{\star n} = \psi^{\star (n-1)} \star \psi \in CB\big(\mathsf{B}; B(\widehat{\mathsf{k}}^{\otimes n})\big) \quad (n \in \mathbb{N}).$$

As usual we are viewing $B(\widehat{\mathsf{k}}^{\otimes n})$ as the multiplier algebra of $K(\widehat{\mathsf{k}}^{\otimes n})$ here, and invoking the remark containing (1.3), to ensure meaning for the convolutions.

We need the following block matrices, on a Hilbert space of the form $\widehat{\mathsf{H}}$:

$$\mathcal{S}_h := \begin{bmatrix} h^{-1/2} & \\ & I_{\mathsf{H}} \end{bmatrix}, \quad h > 0,$$

and write Σ_h for the map $X \mapsto \mathcal{S}_h X \mathcal{S}_h$ on $B(\widehat{\mathsf{H}})$. Such a conjugation provides the correct scaling for quantum random-walk approximation ([LiP]).

Theorem 2.1. *Let $\varphi \in CB(\mathsf{B}; B(\widehat{\mathsf{k}}))$. Suppose that there is a family of maps $(\psi^{(h)})_{0<h\leq C}$ in $CB(\mathsf{B}; B(\widehat{\mathsf{k}}))$ for some $C > 0$, satisfying*

$$\big\|\varphi - \Sigma_h \circ \big(\psi^{(h)} - \iota_{\widehat{\mathsf{k}}} \circ \epsilon\big)\big\|_{\mathrm{cb}} \to 0 \text{ as } h \to 0^+.$$

Then the convolution iterates $\{\psi_n^{(h)} := (\psi^{(h)})^{\star n} : n \in \mathbb{Z}_+\}$ $(0 < h < C)$ satisfy

$$\sup_{t \in [0,T]} \Big\| l_{t,\varepsilon}^{\varphi} - \Xi_{[t/h],\varepsilon}^{(h)} \circ \psi_{[t/h]}^{(h)} \Big\|_{\mathrm{cb}} \to 0 \text{ as } h \to 0^+,$$

for all $T \in \mathbb{R}_+$ and $\varepsilon \in \mathcal{E}$.

Proof. Set $\phi := (\mathrm{id} \otimes \varphi) \circ \Delta$ and $\Psi^{(h)} := (\mathrm{id} \otimes \psi^{(h)}) \circ \Delta$. Denote the enveloping von Neumann algebra of B by $\overline{\mathsf{B}}$ and let $\overline{\phi}, \overline{\Psi^{(h)}} \in CB_\sigma\big(\overline{\mathsf{B}}; B(\widehat{\mathsf{k}})\big)$ denote respectively the normal extensions of ϕ and $\Psi^{(h)}$ to $\overline{\mathsf{B}}$. Similarly let $\overline{\epsilon} : \overline{\mathsf{B}} \to \mathbb{C}$ denote the normal extension of the counit.

As the maps transforming φ into $\overline{\phi}$ and $\psi^{(h)}$ into $\overline{\Psi^{(h)}}$ are complete isometries (by Proposition 2.1 and remarks after Theorem 1.2 in [LS$_2$]), we have

$$\big\|\big(\mathrm{id}_{\overline{\mathsf{B}}} \overline{\otimes} \Sigma_h\big) \circ \big(\Psi^{(h)} - \iota_{\widehat{\mathsf{k}}}\big) - \overline{\phi}\big\|_{\mathrm{cb}} = \big\|\Sigma_h \circ \big(\psi^{(h)} - \iota_{\widehat{\mathsf{k}}} \circ \epsilon\big) - \varphi\big\|_{\mathrm{cb}}$$

which tends to 0 as $h \to 0^+$. Therefore, by Theorem 7.6 of [Be$_2$], it follows that

$$\sup_{t\in[0,T]} \Big\| \big(\mathrm{id}_{\overline{\mathsf{B}}} \,\overline{\otimes}\, \Xi^{(h)}_{[t/h],\varepsilon}\big) \circ \overline{\Psi^{(h)}_{[t/h]}} - k^{\overline{\phi}}_{t,\varepsilon} \Big\|_{\mathrm{cb}} \to 0 \text{ as } h \to 0^+,$$

where $\overline{\Psi^{(h)}_n} := (\overline{\Psi^{(h)}})^{\bullet n}$ and $k^{\overline{\phi}}$ denotes the 'standard' QS cocycle generated by $\overline{\phi}$, that is, the unique weakly regular weak solution of the QS differential equation

$$dk_t = k_t \circ d\Lambda_{\overline{\phi}}(t), \quad k_0 = \iota_{\mathcal{F}}$$

(see [L]). By the results of Section 4 of [LS$_2$] $l^{\varphi}_{t,\varepsilon} = \overline{l}^{\varphi}_{t,\varepsilon}|_{\mathsf{B}}$ where $\overline{l}^{\varphi}_{t,\varepsilon} = (\overline{\epsilon \otimes} \,\mathrm{id}) \circ k^{\overline{\phi}}_{t,\varepsilon}$ ($t \in \mathbb{R}_+, \varepsilon \in \mathcal{E}$). The result therefore follows from the easily checked identity $(\overline{\epsilon\otimes}\,\mathrm{id}) \circ \overline{\Psi^{(h)}_n}|_{\mathsf{B}} = \psi^{(h)}_n$ ($n \in \mathbb{Z}_+$). $\square$

For the next two propositions coproducts play no role. Recall Theorem 1.1 on the automatic implementability of χ-structure maps.

Proposition 2.2. *Let (A, χ) be a C^*-algebra with character and let $\varphi : \mathsf{A} \to B(\widehat{\mathsf{h}})$ be a χ-structure map. Letting (π, ξ) be an implementing pair for φ, set*

$$U^{(h)}_\xi := \begin{bmatrix} c_{h,\xi} & -s^*_{h,\xi} \\ s_{h,\xi} & c_{h,\xi} Q_\xi + Q^\perp_\xi \end{bmatrix}, \text{ for } h > 0 \text{ such that } h\|\xi\|^2 \le 1,$$

where

$$c_{h,\xi} := \sqrt{1 - s^*_{h,\xi} s_{h,\xi}} = \sqrt{1 - h\|\xi\|^2}, s_{h,\xi} = h^{1/2}|\xi\rangle, \text{ and } Q_\xi := P_{\mathbb{C}\,\xi}.$$

Then the following hold.

(a) *Each $U^{(h)}_\xi$ is a unitary operator on $\widehat{\mathsf{h}}$.*

(b) *The family of *-representations*
$\rho^{(h)} = \widehat{\pi}^{(h)}_\xi : \mathsf{B} \to B(\widehat{\mathsf{h}})$ ($h > 0$, $h\|\xi\|^2 \le 1$), defined by

$$\widehat{\pi}^{(h)}_\xi(b) := U^{(h)\,*}_\xi (\chi \oplus \pi)(b) U^{(h)}_\xi$$

satisfies

$$\varphi - \Sigma_h \circ \big(\rho^{(h)} - \iota_{\widehat{\mathsf{h}}} \circ \chi\big) = \frac{h}{1 + c_{h,\xi}}\,\varphi_1 - \frac{h^2}{(1 + c_{h,\xi})^2}\,\varphi_2 \tag{2.1}$$

for some completely bounded maps $\varphi_1, \varphi_2 : \mathsf{A} \to B(\widehat{\mathsf{h}})$ independent of h.

(c) *Each *-representation $\rho^{(h)}$ is nondegenerate if (and only if) π is.*

Proof. For the proof we drop the subscripts ξ, set

$$Q = Q_\xi,\ c_h = c_{h,\xi},\ s_h = s_{h,\xi} \text{ and put } d_h := c_h - 1.$$

(a) This is evident from the identities

$$c^*_h = c_h,\ c^2_h + s^*_h s_h = 1,\ s^*_h Q^\perp = 0 \text{ and } s_h s^*_h = \big(1 - c^2_h\big)Q.$$

(b) Set $\nu = \pi - \iota_{\mathsf{h}} \circ \chi$ and $\gamma = \omega_\xi \circ \nu$ so that φ has block matrix form (1.2). Then, noting the identities

$$d_h = -h\big(1 + c_h\big)^{-1}\|\xi\|^2, \quad \|\xi\|^2 Q = |\xi\rangle\langle\xi|, \quad c_h Q + Q^\perp = d_h Q + I_{\mathsf{h}},$$

we have

$$\begin{aligned}
&\rho^{(h)}(a) - \chi(a) I_{\widehat{\mathsf{h}}} \\
&= U_\xi^{(h)*} \begin{bmatrix} 0 & \\ & \nu(a) \end{bmatrix} U_\xi^{(h)} \\
&= \begin{bmatrix} 0 & h^{1/2}\langle\xi| \\ 0 & d_h Q + I_{\mathsf{h}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ h^{1/2}\nu(a)|\xi\rangle & d_h \nu(a) Q_\xi + \nu(a) \end{bmatrix} \\
&= \begin{bmatrix} h\gamma(a) & h^{1/2}\langle\xi|\nu(a)\big[d_h Q + I_{\mathsf{h}}\big] \\ h^{1/2}\big[d_h Q + I_{\mathsf{h}}\big]\nu(a)|\xi\rangle & d_h^2 Q\nu(a)Q + d_h\big(Q\nu(a) + \nu(a)Q\big) + \nu(a) \end{bmatrix} \\
&= \begin{bmatrix} h^{1/2} & \\ & I_{\mathsf{h}} \end{bmatrix} \Big(\varphi(a) - h\big(1+c_h\big)^{-1}\varphi_1(a) + h^2\big(1+c_h\big)^{-2}\varphi_2(a)\Big) \begin{bmatrix} h^{1/2} & \\ & I_{\mathsf{h}} \end{bmatrix}
\end{aligned}$$

where

$$\varphi_1 = \begin{bmatrix} 0 & \gamma(\cdot)\langle\xi| \\ \gamma(\cdot)|\xi\rangle & X\nu(\cdot) + \nu(\cdot)X \end{bmatrix} \text{ and } \varphi_2 = \gamma(\cdot)\begin{bmatrix} 0 & \\ & X \end{bmatrix} \text{ for } X = |\xi\rangle\langle\xi|,$$

from which (b) follows.

(c) This is evident from the unitality of each $U_\xi^{(h)}$ and the fact that χ is a character. □

Remarks. (i) Both $U_\xi^{(h)}$ and $\widehat{\pi}_\xi^{(h)}$ are norm-continuous in h; they converge to $I_{\widehat{\mathsf{h}}}$ and $\chi \oplus \pi$ respectively as $h \to 0^+$.

(ii) For the simplest class of χ-structure map, namely

$$\varphi = \begin{bmatrix} 0 & \\ & \nu \end{bmatrix}, \quad \text{where } \nu = \pi - \iota_{\mathsf{h}} \circ \chi \text{ for a *-homomorphism } \pi : \mathsf{A} \to B(\mathsf{h}),$$

$U_0^{(h)} = I$ and $\widehat{\pi}_0^{(h)} = (\chi \oplus \pi)$ so

$$\widehat{\pi}_0^{(h)} = \varphi + \iota_{\widehat{\mathsf{h}}}, \text{ for all } h > 0.$$

(iii) Unwrapping $\widehat{\pi}_\xi^{(h)}(a)$:

$$\begin{bmatrix} \chi(a) + h\gamma(a) & s_{h,\xi}^*\big(\nu(a) - \frac{h\gamma(a)}{1+c_{h,\xi}} I_{\mathsf{h}}\big) \\ \big(\nu(a) - \frac{h\gamma(a)}{1+c_{h,\xi}} I_{\mathsf{h}}\big)s_{h,\xi} & \pi(a) - \frac{h}{1+c_{h,\xi}}\big(X\nu(a) + \nu(a)X\big) + \frac{h^2\gamma(a)}{(1+c_{h,\xi})^2} X \end{bmatrix}$$

reveals the vector-state realisation

$$\omega_{e_0} \circ \widehat{\pi}_\xi^{(h)} = \omega_{\Omega_\xi^{(h)}} \circ (\chi \oplus \pi), \text{ where } \Omega_\xi^{(h)} := U_\xi^{(h)} e_0 \text{ and } e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \widehat{\mathsf{h}},$$

for the state $\chi + h\gamma = \chi + h\|\xi\|^2\big(\omega_{\xi'} \circ \pi - \chi\big)$ where ξ' equals $\|\xi\|^{-1}\xi$, or 0 if $\xi = 0$. Indeed, finding such a representation was the strategy of proof in [FrS].

(iv) This remark will be used in the proof of Theorem 2.4. If instead of being a χ-structure map, φ is given by

$$\begin{bmatrix} \langle\xi| \\ D^* \end{bmatrix} \nu(\cdot) \begin{bmatrix} |\xi\rangle & D \end{bmatrix} \text{ where } \nu := \pi - \iota_{\mathsf{H}} \circ \chi, \tag{2.2}$$

for a nondegenerate representation $\pi : \mathsf{A} \to B(\mathsf{H})$, vector $\xi \in \mathsf{H}$ and isometry $D \in B(\mathsf{h}; \mathsf{H})$ then, replacing the unitaries $U_\xi^{(h)}$ by the isometries $V_{\xi,D}^{(h)} := U_\xi^{(h)} \left[\begin{smallmatrix} 1 & \\ & D \end{smallmatrix}\right] \in$

$B(\widehat{\mathsf{h}};\widehat{\mathsf{H}})$ in the above proof yields a family of completely positive preunital maps $\rho^{(h)} = \rho^{(h)}_{\pi,\xi,D}$ $(0 < h,\ h\|\xi\|^2 \le 1)$, defined by

$$\psi^{(h)}_{\pi,\xi,D} := V^{(h)}_{\xi,D}{}^{*}(\chi \oplus \pi)(\cdot)V^{(h)}_{\xi,D}$$

satisfying (2.1), with completely bounded maps

$$\varphi_1 = \begin{bmatrix} 0 & \gamma(\cdot)\langle\eta| \\ \gamma(\cdot)|\eta\rangle & Y^*\nu(\cdot)D + D^*\nu(\cdot)Y \end{bmatrix} \text{ and } \varphi_2 = \gamma(\cdot)\begin{bmatrix} 0 & \\ & X \end{bmatrix}$$

where now $Y = |\xi\rangle\langle\eta|$, for $\eta = D^*\xi \in \mathsf{h}$, but X is still equal to $|\xi\rangle\langle\xi|$.

Proposition 2.3. *Let (A,χ) be a C^*-algebra with character and let $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$. Suppose that $\overline{\varphi}(1) \le 0$ and φ is expressible in the form*

$$\varphi_1 - \varphi_2 \text{ where } \varphi_1 \in CP(\mathsf{A}; B(\widehat{\mathsf{k}})) \text{ and } \varphi_2 = \chi(\cdot)\big(\Delta^{\mathrm{QS}} + |\zeta\rangle\langle e_0| + |e_0\rangle\langle\zeta|\big), \quad (2.3)$$

for a vector $\zeta \in \widehat{\mathsf{k}}$. Then there is $C > 0$ and a family of completely positive contractions $\big(\phi^{(h)} : \mathsf{A} \to B(\widehat{\mathsf{k}})\big)_{0<h\le C}$, such that

$$\big\|\varphi - \Sigma_h \circ \big(\phi^{(h)} - \iota_{\widehat{\mathsf{h}}} \circ \chi\big)\big\|_{\mathrm{cb}} \to 0 \text{ as } h \to 0^+. \quad (2.4)$$

Proof. It follows from Proposition 4.3 and Theorem 4.4 of [S], and their proofs, that there is a Hilbert space h containing k and a χ-structure map $\theta : \mathsf{A} \to B(\widehat{\mathsf{h}})$ such that φ is the compression of θ to $B(\widehat{\mathsf{k}})$. By Proposition 2.2 b, there is some $C > 0$ and a family of *-homomorphisms $\big(\rho^{(h)} : \mathsf{A} \to B(\widehat{\mathsf{h}})\big)_{0<h\le C}$ satisfying

$$\big\|\theta - \Sigma_h \circ \big(\rho^{(h)} - \iota_{\widehat{\mathsf{h}}} \circ \chi\big)\big\|_{\mathrm{cb}} \to 0 \text{ as } h \to 0^+.$$

It follows that (2.4) holds for the compressions $\phi^{(h)}$ of $\rho^{(h)}$ to $B(\widehat{\mathsf{k}})$, which are manifestly completely positive and contractive. □

Combining the above results we obtain the following discrete approximation theorem for QS convolution cocycles.

Theorem 2.4. *Let l be a Markov-regular, completely positive, contractive quantum stochastic convolution cocycle on a counital C^*-bialgebra B. Then there is some $C > 0$ and a family of completely positive contractions $\big(\psi^{(h)} : \mathsf{B} \to B(\widehat{\mathsf{k}})\big)_{0<h\le C}$, such that the convolution iterates $\big(\psi^{(h)}_n := (\psi^{(h)})^{\star n}\big)_{n\in\mathbb{Z}_+}$ satisfy*

$$\sup_{t\in[0,T]} \Big\| l_{t,\varepsilon} - \Xi^{(h)}_{[t/h],\varepsilon} \circ \psi^{(h)}_{[t/h]} \Big\|_{\mathrm{cb}} \to 0 \text{ as } h \to 0^+,$$

*for all $T \in \mathbb{R}_+$ and $\varepsilon \in \mathcal{E}$. Moreover if l is *-homomorphic, and/or preunital, then each $\psi^{(h)}$ may be chosen to be so too.*

Proof. By Theorem 5.2 a of [LS$_2$] we know that $l = l^{\varphi}$ for some $\varphi \in CB(\mathsf{B}; B(\widehat{\mathsf{k}}))$ which has a decomposition of the form (2.3), with $\chi = \epsilon$. The first part therefore follows from Proposition 2.3 and Theorem 2.1. If l is preunital then φ may be expressed in the form (2.2) and so, by the remark containing (2.2), it follows that the completely positive maps $\psi^{(h)}$ may be chosen to be preunital.

Now suppose that l is *-homomorphic. Then, by Theorem 5.2 c of [LS$_2$], φ is an ϵ-structure map and so, by Theorem 1.1, φ has an implementing pair (π,ξ) with π nondegenerate if l is. It therefore follows from Proposition 2.2 that the maps $\psi^{(h)}$ may be chosen to be *-homomorphic — and also nondegenerate if the cocycle l is nondegenerate. This completes the proof. □

Using the standard language of quantum Lévy processes the last statement of the above theorem can be rephrased as follows.

Corollary 2.5. *Every Markov-regular quantum Lévy process on a multiplier C^*-bialgebra can be approximated (in its Fock space realisation, with respect to the pointwise-strong operator topology) by quantum random walks.*

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